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A PERFECT MORSE FUNCTION ON PLANAR POLYGON SPACES

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Abstract

Let $M_{n,r}$ be the moduli space of planar polygons with n vertices, whose side lengths are $1, \ldots, 1$ and r, with $0 < r \in \mathbb{R}$. We construct a perfect Morse function on $M_{n,r}$ for some (n, r).

1. Introduction and Statement of the Results

Throughout this paper, a Morse function is intended to mean a function, whose critical points are non-degenerate. We do not consider a function, whose critical points form submanifolds.

In the celebrated paper [9], Smale proved that if M is a simply connected closed manifold of dimension > 5, then there exists a Morse function f on M with the minimal number of critical points consistent with the homology structure. Let us call such f a *perfect* Morse function. (When $H_*(M; \mathbb{Z})$ are torsion free, a perfect Morse function is a function for which Morse inequalities are equalities.)

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One made efforts to generalize Smale's result to the case that M is not simply connected. For example, Šarko [8] proved that a perfect Morse function exists when $\pi_1(M) = \mathbb{Z}$. But a complete answer for general M is not known.

The purpose of this paper is to study the problem when M is the moduli space of closed planar polygonal curves, whose side lengths are $1, \ldots, 1$ and r, i.e., one is different from others.

We will focus our attention to the moduli space

$$M_{n,r} = \{(z_1, \ldots, z_{n-1}) \in (S^1)^{n-1}; \sum_{i=1}^{n-1} z_i = r, S^1 \subset \mathbb{C}\}, \tag{1.1}$$

for $5 \leq n \in \mathbb{N}$ and $r \in \mathbb{R}$. Here, $z_i \in S^1 \subset \mathbb{C}$ denote the unit vectors in the directions of the sides of a polygon.

From a different point of view, $M_{n,r}$ is the configuration space of a planar linkage, a planar mechanism consisting of n bars of length $1, \ldots, 1$ and r connected by revolving joints. Such mechanisms play an important role in robotics, where they describe closed kinematic chains and are used widely as elementary parts of more complicated mechanisms. We can apply the knowledge of the topology of $M_{n,r}$ in designing control programs and motion planning algorithms for mechanisms.

The moduli spaces $M_{n,r}$ of planar polygonal linkages were studied extensively by many mathematicians; we will mention Thurston and Weeks [10], Hausmann [2], [3], Kapovich and Millson [7] and others. We recall the results below.

Since $M_{n,n-1} = \{\text{one point}\}$, we assume that $0 < r < n-1$. We say that r is in a wall, if and only if $n - r$ is odd, and otherwise, we say that r is in a chamber. It is known that $M_{n,r}$ is a manifold, if and only if r is in a chamber. (See [2], [7].) Moreover, if r_1 and r_2 belongs to the same connected component of a chamber, then M_{n,r_1} and M_{n,r_2} are diffeomorphic. Hence, from now on, we assume that $n - r$ is even and write k for r.

Now, let us consider the question whether there exists a perfect Morse function on $M_{n,k}$. Smale's result does not guarantee the existence because

$$\pi_1(M_{n,k}) = \begin{cases} 0, & \text{if } k = n-2, \\ \pi_1(\Sigma_4), & \text{if } (n, k) = (5, 1), \\ F_{n-1}, & \text{if } k = n-4 \text{ and } n \geq 6, \\ \mathbb{Z}^{n-1,} & \text{if } k \leq n-6, \end{cases}$$

where Σ_4 denotes a connected closed orientable surface of genus 4 and F_{n-1} denotes the free abelian group of rank $n-1$.

About $M_{n,n-2}$, the question is easy. Indeed, if we define a function $f : M_{n,n-2} \to \mathbb{R}$ by $f(z_1, \ldots, z_{n-1}) = y_1$, then f has two critical points. Hence by Reeb's theorem, $M_{n,n-2}$ is homeomorphic to S^{n-3}.

The next case is $M_{n,n-4}$. It is proved in [4] and [10] that $M_{5,1}$ is diffeomorphic to Σ_4. To prove this, [4] constructed a Morse function $f : M_{5,1} \to \mathbb{R}$ by assigning a pentagon to its oriented area. The number of critical points of f of index i is 2, 10 or 2 according as $i = 0, 1$ or 2. Hence f is not a perfect Morse function.

Later, it was proved in [3] that $M_{n,n-4} \cong \underset{n-1}{\#}(S^1 \times S^{n-4})$. Hence, our work contains an explicit description of a perfect Morse function on such a connected sum via polygon spaces.

We define a function $f_n : M_{n,n-4} \to \mathbb{R}$ by

$$f_n(z_1, \ldots, z_{n-1}) = y_1 + y_2 + 2y_3 + 2y_4.$$

Theorem A. *About* f_5, *the following results hold*:

(i) *There is one critical point of index* 0. *This is a pentagon* $(z_1, \ldots, z_4) \in M_{5,1}$ *such that* $z_1 = z_2$, $z_3 = z_4$, *and* $y_1 > 0$.

(ii) *There are* 8 *critical points of index* 1. *They are pentagons, which satisfies the following* (a) *or* (b): (a) z_1 *and* z_2 *are pure imaginary and* $z_1 + z_2 = 0$; (b) z_3 *and* z_4 *are pure imaginary and* $z_3 + z_4 = 0$.

(*If* $z_1 + z_2 = 0$, *then we have two choices for* (z_3, z_4): *They are two edges of an equilateral triangle in the upper or lower half-plane. The choices for the case* $z_3 + z_4 = 0$ *are considered similarly*.)

(iii) *The critical point of index* 2 *is obtained from* (i) *by reflection about the real axis*.

Remark 1.1. Note that $f_5(z_1, \ldots, z_4) = y_3 + y_4$. More generally, we define a function $\widetilde{f}_n : M_{n,n-4} \to \mathbb{R}$ by

$$\widetilde{f}_n(z_1, \ldots, z_{n-1}) = \sum_{i=1}^{s} y_i,$$

where $s = \lfloor (n-1)/2 \rfloor$. Then $\widetilde{f}_n$ is also a perfect Morse function for all $n \neq 6$. But $\widetilde{f}_6$ is not a Morse function.

Let S^{n-1} denote the $(n-1)$-fold direct product of $S := \{-1, 1\}$. For $w = (w_1, \ldots, w_{n-1}) \in S^{n-1}$, we set $\widetilde{w} := |\{i; w_i = -1\}|$, where $|-|$ stands for the cardinality.

Theorem B. (i) *For* $n \geq 6$, *each critical point of* f_n *corresponds to an element* $w \in S^{n-1}$ *such that* $\widetilde{w} = 0, 1, n-2$ *or* $n-1$.

(ii) *For* $n \geq 6$, *the index of the critical point, which corresponds to* w *is given by*

$$\begin{cases} \widetilde{w}, & \text{if } \widetilde{w} = 0, 1, \\ \widetilde{w} - 2, & \text{if } \widetilde{w} = n-2, n-1. \end{cases}$$

(iii) *For* $n \geq 8$, *the shape of an n-gon* $(z_1, \ldots, z_{n-1})$, *which corresponds to* $w = (w_1, \ldots, w_{n-1}) \in S^{n-1}$ *is given as follows*:

(a) *If* $\{i, j\}$ *is* $\{1, 2\}$, $\{3, 4\}$ *or a subset of* $\{5, \ldots, n-1\}$, *then* $w_i = w_j$ *corresponds to* $z_i = z_j$ *and* $w_i = -w_j$ *corresponds to* $z_i = -z_j$.

(b) *If* $\tilde{w} = 0$ *or* 1, *then* $y_1 + y_2 + y_3 + y_4 < 0$. *Moreover,* $w_i = 1$ *corresponds to* $x_i > 0$.

(c) *If* $\tilde{w} = n-2$ *or* $n-1$, *then* $y_1 + y_2 + y_3 + y_4 > 0$. *Moreover,* $w_i = -1$ *corresponds to* $x_i > 0$.

Example 1.2. Let $z^1 \in M_{8,4}$ be the critical point, which corresponds to $w = (1, 1, 1, 1, 1, 1, -1)$. The index of z^1 is 1. We draw z^1 in Figure 1. Note that Theorem B (iii) holds.

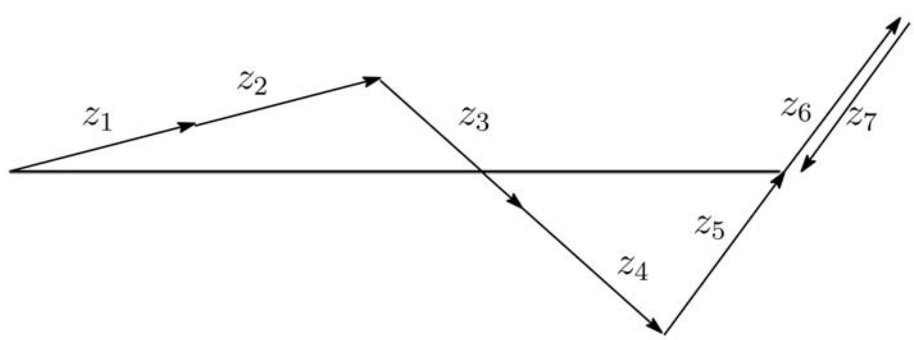

Figure 1. The critical point z^1.

On the other hand, for $n = 6$ and 7, there are exceptional cases for Theorem B (iii) (b) and (c). See Remark 3.1 for more details.

Summarizing Theorems A and B, we have the following:

Theorem C. *For* $n \geq 5$, f_n *is a perfect Morse function.*

Next, we define a function $g_{n,k} : M_{n,k} \to \mathbb{R}$ by

$$g_{n,k}(z_1, \ldots, z_{n-1}) = \sum_{i=1}^{n-1} i\, y_i.$$

Theorem D. *Assume that* (n, k) (*where* $n - k$ *is even*) *satisfies the following* (a) *or* (b): (a) $2k + 3 \geq n$; (b) $(n, k) = (11, 3), (12, 4)$ *or* $(16, 6)$. *Then*, *the following results hold*:

(i) *Each critical point of* $g_{n,k}$ *corresponds to an element* $w \in S^{n-1}$ *such that*

$$0 \leq \widetilde{w} \leq (n - k - 2)/2 \quad \textit{or} \quad (n + k)/2 \leq \widetilde{w} \leq n - 1.$$

(ii) *The index of the critical point*, *which corresponds to w is given by*

$$\begin{cases} \widetilde{w}, & \textit{if } 0 \leq \widetilde{w} \leq (n - k - 2)/2, \\ \widetilde{w} - 2, & \textit{if } (n + k)/2 \leq \widetilde{w} \leq n - 1. \end{cases}$$

Theorem E. *If* (n, k) *satisfies Theorem* D (a) *or* (b), *then* $g_{n,k}$ *is a perfect Morse function.*

Remark 1.3. (i) In fact, $g_{n,k}$ is a Morse function for all (n, k) with $n - k$ even. See also Remark 4.1.

(ii) The author does not know whether $M_{n,r}$ other than Theorem D (a) and (b) has a perfect Morse function. See also Remark 4.2.

This paper is organized as follows. In Section 2, we recall the homology of $M_{n,k}$. Then we recall a lemma, which is used to determine critical points and indices of a function. In Section 3, we prove Theorems A, B, and C. In Section 4, we prove Theorems D and E.

2. Preliminaries

Theorem 2.1 ([1], [6])**.** *For all* (n, k) *with* $n - k$ *even,* $H_*(M_{n,k}; \mathbb{Z})$ *are torsion free and the Poincaré polynomial is given as follows*:

$$PS(M_{n,k}) = \sum_{i=0}^{(n-k-2)/2} \binom{n-1}{i} t^i + \sum_{i=(n+k-4)/2}^{n-3} \binom{n-1}{i+2} t^i.$$

Theorems A, B, and D are proved by using the following well-known lemma. (See, for example, [5].)

Lemma 2.2. *Let* $\phi_1, \ldots, \phi_r : \mathbb{R}^d \to \mathbb{R}$ *be smooth functions. We set*

$$M = \{p \in \mathbb{R}^d;\ \phi_1(p) = \cdots = \phi_r(p) = 0\}.$$

Assume that $(\operatorname{grad}\phi_1)_p, \ldots, (\operatorname{grad}\phi_r)_p$ *are linearly independent for all* $p \in M$. *For a smooth function* $\phi : \mathbb{R}^d \to \mathbb{R}$, *we set* $f := \phi|_M : M \to \mathbb{R}$. *Then, the following results hold:*

(i) *A point* $p_0 \in M$ *is a critical point of f, if and only if* $(\operatorname{grad}\phi)_{p_0}$ *is a linear combination of* $(\operatorname{grad}\phi_1)_{p_0}, \ldots, (\operatorname{grad}\phi_r)_{p_0}$:

$$(\operatorname{grad}\phi)_{p_0} = \alpha_1(\operatorname{grad}\phi_1)_{p_0} + \cdots + \alpha_r(\operatorname{grad}\phi_r)_{p_0}, \quad \alpha_i \in \mathbb{R}.$$

(ii) *Let* $p_0 \in M$ *be a critical point of f and* $P : \mathbb{R}^d \to T_{p_0}M$ *be the orthogonal projection. Let H be the Hessian of the function* $\phi - \sum_{i=1}^r \alpha_i\phi_i$ *at* p_0. *We define* $A := PHP$. *Then* p_0 *is non-degenerate, if and only if* rank $A = d - r$. *Moreover, the index of f at* p_0 *equals to the number of negative eigenvalues of A.*

Remark 2.3. We can compute P as follows. Let J be the Jacobian matrix of $(\phi_1, \ldots, \phi_r)$ at p_0. Then, we have $P = E_d - {}^tJ(J \cdot {}^tJ)^{-1}J$, where E_d is the identity matrix.

3. Proof of Theorems A, B and C

Setting $z_i = \cos\theta_i + \sqrt{-1}\sin\theta_i$, we use Lemma 2.2 for

$$\phi_1(\theta_1, \ldots, \theta_{n-1}) = \cos\theta_1 + \cdots + \cos\theta_{n-1} - (n-4),$$

$$\phi_2(\theta_1, \ldots, \theta_{n-1}) = \sin\theta_1 + \cdots + \sin\theta_{n-1},$$

and

$$\phi(\theta_1, \ldots, \theta_{n-1}) = \sin\theta_1 + \sin\theta_2 + 2\sin\theta_3 + 2\sin\theta_4.$$

We need to solve an equation

$$\operatorname{grad}\phi = \alpha_1 \operatorname{grad}\phi_1 + \alpha_2 \operatorname{grad}\phi_2 \text{ (for some } \alpha_1, \alpha_2 \in \mathbb{R}), \tag{3.1}$$

under the conditions $\phi_1 = \phi_2 = 0$.

Proof of Theorem B. We claim that for $1 \le i \le n-1$, we can write

$$(\cos\theta_i, \sin\theta_i) = w_i\left(\frac{\alpha_1}{\sqrt{\alpha_1^2 + (\alpha_2 - r_i)^2}}, \frac{\alpha_2 - r_i}{\sqrt{\alpha_1^2 + (\alpha_2 - r_i)^2}}\right), \tag{3.2}$$

where $w_i \in S$, and we set

$$r_i = \begin{cases} 1, & 1 \le i \le 2, \\ 2, & 3 \le i \le 4, \\ 0, & 5 \le i \le n-1. \end{cases}$$

To show this, note that we have in (3.1) that

$$(\alpha_1, \alpha_2) \ne (0, 0), (0, 1), \text{ and } (0, 2).$$

In fact, if $(\alpha_1, \alpha_2) = (0, 0)$, then $\theta_1, \ldots, \theta_4 = \pm\pi/2$. But this contradicts the equation $\phi_1 = \phi_2 = 0$. The other cases can be proved similarly. Hence (3.2) holds.

Substitute (3.2) into the equation $\phi_1 = \phi_2 = 0$. Then direct computations show that there is a solution (α_1, α_2), if and only if $\widetilde{w} = 0, 1, n-2$ or $n-1$. Moreover, the solution is unique for a fixed w. Hence Theorem B (i) holds. We can also prove Theorem B (ii) and (iii) easily. □

Remark 3.1. Let us give exceptional cases to Theorem B (iii) (b) and (c). There is one critical point $z^2 \in M_{6,2}$ of index 1 with $y_1 + y_2 + y_3 + y_4 > 0$. This corresponds to $w = (1, 1, 1, 1, -1)$. We draw this in Figure 2.

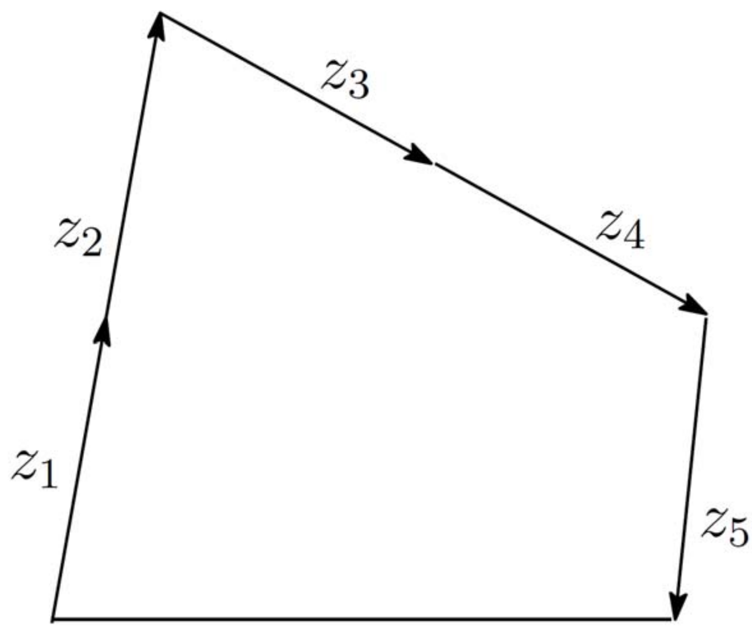

Figure 2. The critical point z^2.

There are two critical points z^3 and $z^4 \in M_{7,4}$ of index 1 with $y_1 + y_2 + y_3 + y_4 = 0$. They correspond to $w = (1, 1, 1, 1, -1, 1)$ and $(1, 1, 1, 1, 1, -1)$, respectively. We draw z^3 in Figure 3. The figure of z^4 is obtained from z^3 by changing only z_5 and z_6 to $-z_5$ and $-z_6$.

We can check easily that z^2, z^3, z^4 and their reflections about the real axis exhaust exceptional cases for Theorem B (iii) (b) and (c).

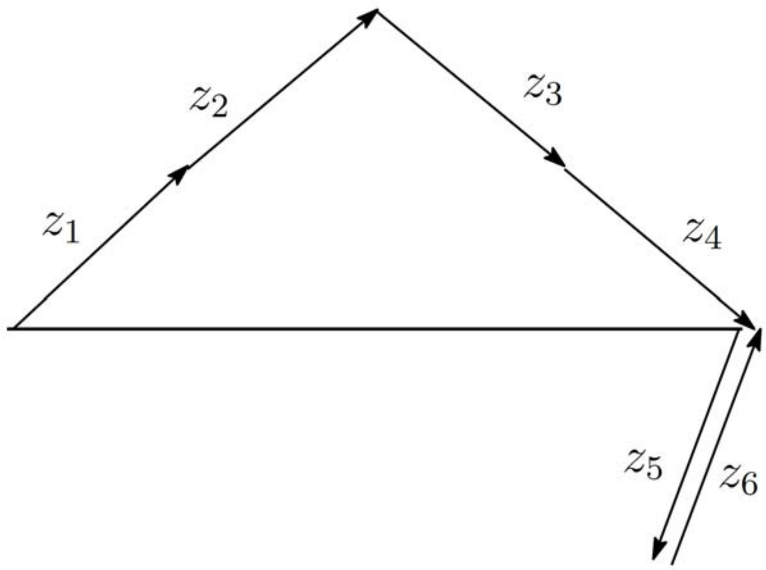

Figure 3. The critical point z^3.

Proof of Theorem A. Consider (3.1). If $(\alpha_1, \alpha_2) \neq (0, 1), (0, 2)$, then we have reflectional two critical points satisfying $z_1 = z_2$ and $z_3 = z_4$. Since the point in $y_1 > 0$ gives the minimum of f_5, the index of the point is 0.

On the other hand, if $(\alpha_1, \alpha_2) = (0, 1)$ or $(0, 2)$, then $(\theta_1, \theta_2) = (\pi/2, -\pi/2)$ or $(\theta_3, \theta_4) = (\pi/2, -\pi/2)$. The indices of these critical points are 1. $\square$

Proof of Theorem C. The corollary is clear from Theorems A and B.

$\square$

4. Proof of Theorems D and E

Proof of Theorem D. We fix (n, k) as in Theorem D (a) or (b). As in Section 3, we set

$$\phi_1(\theta_1, \ldots, \theta_{n-1}) = \cos\theta_1 + \cdots + \cos\theta_{n-1} - k,$$

$$\phi_2(\theta_1, \dots, \theta_{n-1}) = \sin\theta_1 + \cdots + \sin\theta_{n-1},$$

and

$$\phi(\theta_1, \dots, \theta_{n-1}) = \sin\theta_1 + 2\sin\theta_2 + \cdots + (n-1)\sin\theta_{n-1}.$$

Similar to (3.2), the condition for (n, k) implies that we can write

$$(\cos\theta_i, \sin\theta_i) = w_i\left(\frac{\alpha_1}{\sqrt{\alpha_1^2 + (\alpha_2 - i)^2}}, \frac{\alpha_2 - i}{\sqrt{\alpha_1^2 + (\alpha_2 - i)^2}}\right). \tag{4.1}$$

Substitute (4.1) into the equation $\phi_1 = \phi_2 = 0$. Then direct computations show that there is a solution (α_1, α_2), if and only if $\widetilde{w}$ is in the range of Theorem D (i). Moreover, the solution is unique for a fixed w. Hence Theorem D (i) holds. We can also prove Theorem D (ii) easily. □

Proof of Theorem E. Using Theorem 2.1, we see that $g_{n,k}$ is perfect when (n, k) satisfies Theorem D (a) or (b). □

Remark 4.1. The function $g_{n,k}$ is a Morse function for all (n, k) with $n - k$ even. We have the following result: Assume that $g_{n,k}$ is not perfect. Then, the least i such that $\alpha_i(g_{n,k}) > b_i(M_{n,k})$ (where $\alpha_i(g_{n,k})$ is the number of critical points of $g_{n,k}$ of index i and $b_i(M_{n,k})$ is the i-th Betti number of $M_{n,k}$) is given by

$$\begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k = 2, \\ (n-k-2)/2, & \text{if } k \geq 3. \end{cases}$$

In particular, $g_{n,2}$ (for even n) has two local minimum points and only one of them is a global minimum point.

Remark 4.2. The author conjectures that $M_{n,1}$ does not have a perfect Morse function for $n \geq 7$. We give some computational results. For a function $f : M_{n,r} \to \mathbb{R}$, let $\mu_t(f) := \sum_{i=0}^{n-3} \alpha_i(f) t^i$ denote the Morse polynomial.

(i) (a) We have

$$\mu_t(g_{7,1}) = 1 + 9t + 36t^2 + 9t^3 + t^4.$$

(b) We have

$$\mu_t(f_7) = 2 + 8t + 32t^2 + 8t^3 + 2t^4.$$

(ii) We have

$$\mu_t(g_{9,1}) = 1 + 9t + 50t^2 + 154t^3 + 50t^4 + 9t^5 + t^6.$$

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